

# INTEGRAL REPRESENTATION OF LINEAR FUNCTIONALS ON FUNCTION SPACES

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**ABSTRACT.** Let  $A$  be a vector space of real valued functions on a non-empty set  $X$  and  $L : A \rightarrow \mathbb{R}$  a linear functional. Given a  $\sigma$ -algebra  $\mathcal{A}$ , of subsets of  $X$ , we present a necessary condition for  $L$  to be representable as an integral with respect to a measure  $\mu$  on  $X$  such that elements of  $\mathcal{A}$  are  $\mu$ -measurable. This general result then is applied to the case where  $X$  carries a topological structure and  $A$  is a family of continuous functions and naturally  $\mathcal{A}$  is the Borel structure of  $X$ . As an application, short solutions for the full and truncated  $K$ -moment problem are presented. An analogue of Riesz–Markov–Kakutani representation theorem is given where  $C_c(X)$  is replaced with whole  $C(X)$ . Then we consider the case where  $A$  only consists of bounded functions and hence is equipped with sup-norm.

## 1. INTRODUCTION

A positive linear functional on a function space  $A \subseteq \mathbb{R}^X$  is a linear map  $L : A \rightarrow \mathbb{R}$  which assigns a non-negative real number to every function  $f \in A$  that is globally non-negative over  $X$ . The celebrated Riesz–Markov–Kakutani representation theorem states that every positive functional on the space of continuous compactly supported functions over a locally compact Hausdorff space  $X$ , admits an integral representation with respect to a regular Borel measure on  $X$ . In symbols  $L(f) = \int_X f \, d\mu$ , for all  $f \in C_c(X)$ . Riesz’s original result [12] was proved in 1909, for the unit interval  $[0, 1]$ . Then in 1938, Markov extended Riesz’s result to some non-compact spaces [9]. Later in 1941, Kakutani [8] proved the result for compact Hausdorff spaces. Some authors such as Pedersen [11, §6.1], take the reverse approach. Instead of defining an integral operator based on a pre-existing measure, he starts with a special type linear functionals. He calls the class of positive linear functionals on  $C_c(X)$ , the “Radon integrals” and then fixing such a functional, defines integrable functions and the measure which corresponds to  $L$ . This approach has introduced and studied by P. J. Daniell in 1918 [4] and nowadays is known as Daniell integral theory.

Perhaps one of the most well-known consequences of Riesz’s result is its application to solve the classical moment problem, given by Haviland in 1936. Haviland published series of two papers [6, 7], and gave a complete solution for the  $K$ -moment problem. Let  $L : \mathbb{R}[X] = \mathbb{R}[X_1, \dots, X_n] \rightarrow \mathbb{R}$  be a linear functional on the space of real polynomials in  $n$  variables and  $K \subseteq \mathbb{R}^n$  a closed set. The classical  $K$ -moment

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problem asks when  $L$  admits an integral representation with respect to a Borel measure supported on  $K$ . Haviland proved that such a measure exists if and only if  $L$  maps every non-negative polynomial on  $K$  to a non-negative real number. Let us demystify the connection and similarity between these two results.

Fix a subspace  $A \subseteq \mathbb{R}^X$  where  $X$  is a locally compact Hausdorff space and  $K \subseteq X$ , a closed set. The set of all functions  $f \in A$  that are non-negative on  $K$  will be denoted by  $\text{Psd}_A(K)$ :

$$\text{Psd}_A(K) := \{f \in A : f(x) \geq 0 \quad \forall x \in K\}.$$

Let  $L : A \rightarrow \mathbb{R}$  be a linear functional. For  $A = C_c(X)$ , Riesz–Markov–Kakutani’s result is the following statement:

**Theorem 1.1.** *There exists a Borel measure on  $X$  representing  $L$  if and only if  $L(\text{Psd}_{C_c(X)}(X)) \subseteq [0, \infty)$ .*

Also, for  $K \subseteq \mathbb{R}^n = X$  and  $A = \mathbb{R}[X]$ , Haviland’s theorem is the following:

**Theorem 1.2.** *There exists a Borel measure on  $K$  representing  $L$  if and only if  $L(\text{Psd}_{\mathbb{R}[X]}(K)) \subseteq [0, \infty)$ .*

The proof of Haviland’s theorem relies on Riesz–Markov–Kakutani’s result and involves the following steps:

- (1) Extend  $L$  positively to the algebra  $\check{A}$  consisting of all continuous functions, bounded by a polynomial on  $K$ .
- (2) Showing that  $\check{A}$  contains  $C_c(K)$ .
- (3) Applying Riesz–Markov–Kakutani representation theorem to the restriction of the map from step (1) on  $C_c(K)$ .
- (4) Proving that the measure obtained in step (3) also represents  $L$  on  $\mathbb{R}[X]$ .

Suppose that  $\mu$  is a finite Borel measure on  $X$  and let  $\|\cdot\|_{\mu,1}$  be the seminorm induced by  $\mu$  on  $L_1(\mu)$ . One can show that the characteristic function of every Borel set, can be approximated by compactly supported functions in  $\|\cdot\|_{\mu,1}$ . Therefore  $C_c(X)$  is dense in  $C_c(X) \oplus S_{\mathcal{B}(X)}$  where  $S_{\mathcal{B}(X)}$  is the algebra of the *simple functions*, generated by characteristic functions of all Borel subset of  $X$ . On the other hand, simple functions are dense in  $L_1(\mu)$  which implies the denseness of  $C_c(X)$  in  $L_1(\mu)$ .

In section 2, we generalize this idea to any abstract measure structures on  $X$ . In lemma 2.3(4) we prove an analogue of theorem 1.1 and will use this result to prove theorem 2.5 which is a general version of Haviland’s theorem 1.2.

In section 3, we consider perhaps the most interesting case, where the underlying domain consists of continuous functions. We presents short proofs for results of Choquet (Corollary 3.1) and Marshall (Corollary 3.3). Also we apply the results of section 2 to give a simple solution for truncated moment problem in §3.2. The Riesz–Markov–Kakutani representation theorem is stated for  $C_c(X)$  which easily generalizes to  $C_0(X)$ , the space of functions vanishing at infinity. In §3.3 we investigate the case where the domain is extended to whole  $C(X)$  and show that in this case, the support of representing measure has to be compact set (Theorem 3.8).

In section 4, we focus on subalgebras of  $\ell^\infty(X)$ , the algebra of globally bounded functions. This algebra naturally is equipped with a norm topology, where the norm is defined by  $\|f\|_X = \sup_{x \in X} |f(x)|$ . We prove that every continuous positive functional on a subalgebra  $A$  of  $\ell^\infty(X)$ , admits an integral representation with respect to a unique Radon measure on the Gelfand spectrum  $\mathbf{sp}_{\|\cdot\|_X}(A)$  of  $(A, \|\cdot\|_X)$ .

We also consider the possibility of existence of a representing measure on  $X$ , where  $X$  can be realised as a dense subspace of  $\mathfrak{sp}_{\|\cdot\|_X}(A)$ .

## 2. INTEGRAL REPRESENTATION OF A POSITIVE FUNCTIONAL

In this section we study integral representation of a linear functional on a subalgebra of  $\mathbb{R}^X$ . We also prove a variation of Riesz representation theorem which holds for  $\sigma$ -compact, locally compact spaces and replaces  $C_0(X)$  with  $C(X)$  (Theorem 3.8).

The following result plays a key role in this section.

**Theorem 2.1.** *Let  $W$  be a subspace of an  $\mathbb{R}$ -vector space  $V$  and  $C \subseteq V$  a convex cone. Let  $L : W \rightarrow \mathbb{R}$  be a functional with  $L(W \cap C) \geq 0$ . Then  $L$  admits an extension  $\tilde{L}$  on  $W_C$  such that  $\tilde{L}(W_C \cap C) \geq 0$ . Here*

$$W_C = \{v \in V : \pm v \in C + W\}.$$

*Proof.* It is clear that  $W_C$  is a subspace of  $V$  containing  $W$ . We show that the function  $p(v) = -\sup\{L(w) : w \in W \wedge v - w \in C\}$  which is defined on  $W_C$  is a sublinear function such that  $p|_W = -L$ . To see this note that there are  $w, w' \in W$  and  $c, c' \in C$  such that  $v = w + c = w' - c'$ . Thus  $w' - w = c + c' \in C \cap W$  and hence  $L(w' - w) \geq 0$  or  $L(w) \leq L(w')$ . Therefore the set  $\{L(w) : w \in W \wedge v - w \in C\}$  is non-empty and bounded above. Hence  $p(v)$  exists. Clearly  $p(\lambda v) = \lambda p(v)$ , so it remains to show that  $p(v + v') \leq p(v) + p(v')$ . If  $v - w \in C$  and  $v' - w' \in C$ , then  $(v + v') - (w + w') \in C + C = C$ . Thus  $-p(v) - p(v') \leq -p(v + v')$  or equivalently  $p(v + v') \leq p(v) + p(v')$ . For every  $v \in W$ ,  $0 = v - v \in C$ , therefore  $p(v) \leq -L(v)$ . Also for every  $w \in W$  with  $v - w \in C$ , we have  $L(w) \leq L(v)$ , because  $L(W \cap C) \geq 0$ . Therefore  $-L(v) \leq p(v)$  which proves  $p|_W = -L$ .

Applying Hahn-Banach theorem,  $-L$  admits an extension  $\tilde{L}$  to  $W_C$  such that  $-\tilde{L}(v) \leq p(v)$  on  $W_C$ . For  $c \in C \cap W_C$ ,  $p(c) \leq 0$  and hence  $0 \leq -p(c) \leq \tilde{L}(c)$  as desired (For the original result, see [2, Theorem 34.2]).  $\square$

**Definition 2.2.** For a vector subspace  $A$  of  $\mathbb{R}^X$ ,

- (1) The set of all elements of  $A$  that are positive on  $X$  is denoted by  $\text{Psd}_A(X)$ .

$$\text{Psd}_A(X) = \{a \in A : \forall x \in X \quad a(x) \geq 0\};$$

- (2) The *lattice hull* of  $A$  denote by  $\check{A}$  is defined to be

$$\check{A} := \{f \in \mathbb{R}^X : \exists g \in A \quad |f| \leq |g|\}.$$

$A$  is said to be *lattice complete* if  $\check{A} = A$ .

- (3) Let  $B \subseteq \mathbb{R}^X$  be a vector subspace of  $\mathbb{R}^X$ . For  $f, g \in A$ , we say  $g$  is *dominated by  $f$  with respect to  $B$*  and write  $g \preceq_B f$  if

$$\forall \epsilon > 0 \quad \exists h \in B \quad |g| \leq \epsilon |f| + h.$$

We say  $A$  is  *$B$ -adapted* if  $\forall g \in A \quad \exists f \in A$  such that  $g \preceq_B f$ .

- (4) For an algebra  $\mathcal{A}$  of subsets of  $X$ , we denote by  $S_{\mathcal{A}}$ , the algebra generated by characteristic functions of elements of  $\mathcal{A}^1$ .

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<sup>1</sup> $\chi_Y$  denotes the character function of the set  $Y$  defined by  $\chi_Y(x) = 1$  if  $x \in Y$  and 0 otherwise.

- (5) The set of all (infinite) sums of simple functions that are bounded above by an element of  $A$  in absolute value will be denoted by  $\overline{S_A^A}$ ,

$$\overline{S_A^A} := \left\{ \sum_{i=1}^{\infty} \lambda_i \chi_{X_i} : X_i \in \mathcal{A} \wedge \exists a \in A \mid \sum_{i=1}^{\infty} \lambda_i \chi_{X_i} \leq a \right\}.$$

It is clear that  $\overline{S_A^A}$  is a vector space.

**Lemma 2.3.** *Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of  $X$ ,  $B$  a lattice complete algebra of  $\mathcal{A}$ -measurable functions for which every  $\chi_Y$ ,  $Y \in \mathcal{A}$  is bounded above in  $B$  and  $L : B \rightarrow \mathbb{R}$  a positive functional. Then*

- (1)  $L$  extends positively to a linear functional  $\bar{L} : B + S_A \rightarrow \mathbb{R}$  (and also to  $B + \overline{S_A^B}$ );
- (2) the function  $\rho_L(f) = L(|f|)$  (resp.  $\rho_L = \bar{L}(|f|)$ ) defines a seminorm on  $B$  (resp. on  $B + S_A$ );
- (3) every function  $f \in B$  can be approximated by elements of  $\overline{S_A^B}$  from below, in  $\rho_L$ ;
- (4) if  $B$  is dense in  $(B + S_A, \rho_L)$ , then there exists a measure  $\mu$  on  $(X, \mathcal{A})$  such that

$$\forall f \in \text{Psd}_B(X) \quad L(f) = \int f \, d\mu.$$

*Proof.* (1) Let  $V = B + S_A$  and  $C = \text{Psd}_V(X)$ . It suffices to show that  $V = B + C$ , then the conclusion follows from theorem 2.1. A typical element of  $V$  is of the form  $b + \sum_{i=1}^n \lambda_i \chi_{Y_i}$ ,  $b \in B$ ,  $Y_i \in \mathcal{A}$  and  $\lambda_i \in \mathbb{R}$ , for  $i = 1, \dots, n$ . By assumption, for each  $i = 1, \dots, n$ , there is  $b_i \in B$  with  $\chi_{Y_i} \leq b_i$ . Take  $b' = \sum_{i=1}^n |\lambda_i| b_i$ , then  $b' \pm \sum_{i=1}^n \lambda_i \chi_{Y_i} \in C$  and

$$b \pm \sum_{i=1}^n \lambda_i \chi_{Y_i} = (b - b') + (b' \pm \sum_{i=1}^n \lambda_i \chi_{Y_i}) \in B + C.$$

Therefore  $V = B + C$  as desired. Now let  $V = B + \overline{S_A^B}$  and  $C = \text{Psd}_V(X)$ . For  $b \pm \sum_{i=1}^{\infty} \lambda_i \chi_{X_i}$ , there exists  $b' \in B$  such that  $|\sum_{i=1}^{\infty} \lambda_i \chi_{X_i}| < b'$ , so  $b' \pm \sum_{i=1}^{\infty} \lambda_i \chi_{X_i} \in C$  and

$$b \pm \sum_{i=1}^{\infty} \lambda_i \chi_{X_i} = (b - b') + (b' \pm \sum_{i=1}^{\infty} \lambda_i \chi_{X_i}) \in B + C,$$

and the conclusion follows from theorem 2.1.

(2) This is clear. Since  $B$  is lattice complete, it contains absolute value of its elements (note that absolute value of an  $\mathcal{A}$ -measurable function is  $\mathcal{A}$ -measurable as well). Therefore by positivity of  $L$  we see  $\rho_L(rf) = |r|L(|f|) = |r|\rho_L(f)$  and  $\rho_L(f+g) = L(|f+g|) \leq L(|f|+|g|) = L(|f|) + L(|g|) = \rho_L(f) + \rho_L(g)$ .

(3) Fix  $f \in B$  and  $\epsilon > 0$ . First suppose that  $f(X)$  is bounded. This means that there are  $a < b \in \mathbb{R}$  such that for every  $x \in X$ ,  $a \leq f(x) < b$ . Let  $n \geq 1$  be an integer and define

$$X_k := f^{-1} \left( \left[ a + (k-1) \frac{b-a}{n}, a + k \frac{b-a}{n} \right) \right), \quad k = 1, \dots, n.$$

Each  $X_k$  is  $\mathcal{A}$ -measurable. Take  $\phi = \frac{b-a}{n} \sum_{k=1}^n (k-1) \chi_{X_k}$ , this is clear that  $\phi(x) \leq f(x)$  for all  $x \in X$  and  $0 \leq f(x) - \phi(x) < \frac{b-a}{n}$ . Therefore  $0 \leq \bar{L}(f - \phi) \leq \frac{b-a}{n} \bar{L}(1)$ . Choose  $n$  big enough such that  $\frac{b-a}{n} \bar{L}(1) < \epsilon$ , we will have  $\rho_{\bar{L}}(f - \phi) = \bar{L}(f - \phi) < \epsilon$ .

Now take an arbitrary  $f \in B$  and let  $\{[a_i, b_i]\}_{i \in \mathbb{N}}$  be a partition of  $\mathbb{R}$  and  $\epsilon > 0$ . Since  $B$  is lattice complete and  $|\chi_{[a_i, b_i]} f| \leq |f|$ , we see that  $\chi_{[a_i, b_i]} f \in B$ . For every  $i \in \mathbb{N}$ , there exists  $\phi_i \in S_{\mathcal{A}}$  such that  $\rho_{\bar{L}}(\chi_{[a_i, b_i]} f - \phi_i) < 2^{-i} \epsilon$  and  $\phi_i \leq \chi_{[a_i, b_i]} f$ . Let  $\psi = \sum_{i=1}^{\infty} \phi_i$ , then  $|\psi| \leq |f|$  on  $X$ , so  $\psi \in \overline{S_{\mathcal{A}}}^B$  and

$$\begin{aligned} \rho_{\bar{L}}(f - \psi) &= \rho_{\bar{L}}\left(\sum_{i=1}^{\infty} \chi_{[a_i, b_i]} f - \phi_i\right) \\ &\leq \sum_{i=1}^{\infty} \rho_{\bar{L}}(\chi_{[a_i, b_i]} f - \phi_i) \\ &< \sum_{i=1}^{\infty} 2^{-i} \epsilon \\ &= \epsilon, \end{aligned}$$

as desired.

(4) Let  $\bar{L}$  be the functional from part (1) and  $\{Y_n\}_n \subset \mathcal{A}$  a sequence of pairwise disjoint sets. Then  $Y = \cup_n Y_n \in \mathcal{A}$  and  $\chi_Y = \sum_{n=1}^{\infty} \chi_{Y_n}$ . Since  $\bar{L}$  is positive,  $0 \leq \sum_1^n \bar{L}(\chi_{Y_i}) + \bar{L}(\sum_{n+1}^{\infty} \chi_{Y_i}) = \bar{L}(\chi_Y)$ . The sequence  $(\bar{L}(\sum_{n+1}^{\infty} \chi_{Y_i}))_n$  is positive and decreasing and hence convergent. Thus for any  $\epsilon > 0$ , there exists  $N > 0$  such that for any  $n > m > N$ ,

$$\begin{aligned} \bar{L}(\sum_{m+1}^{\infty} \chi_{Y_i}) - \bar{L}(\sum_{n+1}^{\infty} \chi_{Y_i}) &= \bar{L}(\sum_{m+1}^n \chi_{Y_i}) \\ &= \sum_{m+1}^n \bar{L}(\chi_{Y_i}) \\ &< \frac{\epsilon}{2}, \end{aligned}$$

hence  $\sum_{m+1}^{\infty} \bar{L}(\chi_{Y_i}) < \epsilon$ . Therefore  $\lim_{n \rightarrow \infty} \sum_{n+1}^{\infty} \bar{L}(\chi_{Y_i}) = 0$ . So the set function defined on  $\mathcal{A}$  by  $\mu(A) = \bar{L}(\chi_Y)$  is indeed a measure on  $\mathcal{A}$ .

Now, for any  $\mu$ -measurable function  $f \geq 0$ ,

$$\begin{aligned} \int f \, d\mu &= \sup\{\int \phi \, d\mu : \phi \in S_{\mathcal{A}} \text{ and } \phi \leq f\} \\ &= \sup\{\bar{L}(\phi) : \phi \in S_{\mathcal{A}} \text{ and } \phi \leq f\}. \end{aligned}$$

By density of  $B$  in  $(B + S_{\mathcal{A}}, \rho_{\bar{L}})$  and applying part (3), for any  $f \in B$  we have:

$$\sup\{\bar{L}(\phi) : \phi \in S_{\mathcal{A}} \text{ and } \phi \leq f\} = \sup\{L(g) \in B : g \in B \text{ and } g \leq f\} = L(f),$$

therefore  $L(f) = \int f \, d\mu$ .  $\square$

**Proposition 2.4.** *If  $A$  is an algebra then every positive functional  $L : A \rightarrow \mathbb{R}$  admits a positive extension to  $\check{A}$ .*

*Proof.* Note that for  $g \in A$ ,  $|g| \leq \frac{g^2+1}{2}$ . Therefore, for every  $f \in \check{A}$  there exists  $g \in \text{Psd}_A(X)$  such that  $|f| \leq g$  and hence  $f = -g + (g + f) \in A + \text{Psd}_{\check{A}}(X)$ . Applying theorem 2.1 the conclusion follows.  $\square$

**Theorem 2.5.** *Let  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of  $X$ ,  $A \subseteq \mathbb{R}^X$  an algebra of  $\mathcal{A}$ -measurable functions,  $L : A \rightarrow \mathbb{R}$ , a positive functional and  $B \subseteq \check{A}$  a lattice complete subspace, such that*

- (1)  $A$  is  $B$ -adapted and,
- (2)  $B$  is  $\rho_{\bar{L}}$  dense in  $B + S_{\mathcal{A}}$ .

*Then there is a measure  $\mu$  on  $X$  such that  $(X, \mathcal{A}, \mu)$  is a measure space and*

$$\forall f \in \check{A} \quad \bar{L}(f) = \int f \, d\mu.$$

*Proof.* Since  $L$  is positive by theorem 2.1 and proposition 2.4, it extends positively to  $\tilde{L} : \check{A} \rightarrow \mathbb{R}$ . Abusing the notations, we also denote the extension of  $\tilde{L}|_B$  to  $B + S_A$  by  $\tilde{L}$ . Since  $B$  is dense in  $(B + S_A, \rho_{\tilde{L}})$ , by lemma 2.3, there exists a measure  $\mu$  such that  $(X, \mathcal{A}, \mu)$  is a measure space and  $\tilde{L}(f) = \int f d\mu$  for all  $f \in B + S_A$ . For every  $f \in \check{A}$ ,

$$\begin{aligned} \int f d\mu &= \sup\{\int \phi d\mu : \phi \in \mathcal{A} \text{ and } \phi \leq f\} \\ &= \sup\{\int b d\mu : b \in B \text{ and } b \leq f\} \\ &\leq \tilde{L}(f). \end{aligned}$$

Let  $T(f) = \tilde{L}(f) - \int f d\mu$ , then  $T|_B = 0$ . Since  $A$  is  $B$ -adapted, for every  $g \in \text{Psd}_{\check{A}}(X)$  there exists  $f \in \text{Psd}_{\check{A}}(X)$  such that

$$\forall \epsilon > 0 \exists h \in B \quad g \leq \epsilon f + h.$$

Therefore  $0 \leq T(g) \leq \epsilon T(f) + T(h) = \epsilon T(f)$ . Letting  $\epsilon \rightarrow 0$ , we get  $T(g) = 0$  for all  $g \in \check{A}$  and hence for all  $g \in \check{A}$ ,  $\tilde{L}(g) = \int g d\mu$ .  $\square$

**Remark 2.6.** In theorem 2.5, suppose that  $A \subseteq V \subseteq \mathbb{R}^X$  are just vector spaces and  $L : V \rightarrow \mathbb{R}$  a positive linear functional and there is a lattice complete subspace  $B \subseteq \check{A}$  which is dense in  $(B + S_A, \rho_L)$  and for every  $g \in A$  there exists  $f \in V$  such that for all  $\epsilon > 0$ ,  $|g| \leq \epsilon|f| + h$  for some  $h \in B$ , then the same argument proves that  $L$  admits an integral representation on  $A$ .

### 3. ALGEBRA OF CONTINUOUS FUNCTIONS

Perhaps the most interesting case happens when the function algebra is considered to be a subalgebra of continuous functions with respect to a given topology on  $X$ . A natural question in this case is when the resulting measure is Radon or Borel? The main motivation to study representation of a positive functional on a subalgebra of continuous functions to us is the classical moment problem, where the subalgebra is taken to be polynomials on a subset of  $\mathbb{R}^n$ . In this section we restate a slightly more general version of Haviland's solution for multidimensional moment problem.

**Corollary 3.1** (Choquet [2]). *Let  $X$  be a locally compact Hausdorff space,  $A \subseteq C(X)$  a  $C_c(X)$ -adapted space which separates points of  $X$  and  $L : A \rightarrow \mathbb{R}$  a positive linear functional. Then  $L$  is representable via a positive Borel measure on  $X$ .*

*Proof.* Take  $\mathcal{A} = \mathcal{B}(X)$ , the Borel algebra of  $X$ . Let  $g \in C_c(X)$ , we show that  $g \in \check{A}$ . Since  $A$  separates points of  $X$ , for every  $x \in X$ , there exists  $g_x \in \text{Psd}_A(X)$  such that  $|g(x)| < g_x(x)$  holds on an open neighbourhood  $V_x$  of  $x$ . The family  $(V_x)_{x \in X}$  forms an open cover for  $\text{supp}(g)$  which is compact, hence there exist  $x_1, \dots, x_n$ , such that  $\text{supp}(g) \subseteq \cup_i^n V_{x_i}$ . Therefore  $|g| < \sum_1^n g_{x_i}$  and  $g \in \check{A}$ . Thus  $\tilde{L}$  is positive on  $C_c(X)$  and by Riesz–Markov–Kakutani representation theorem, there exists a positive Borel measure  $\mu$  on  $X$  such that  $\tilde{L}(f) = \int f d\mu$  on  $C_c(X)$  and since  $A$  is  $C_c(X)$ -adapted, theorem 2.5 implies that  $\tilde{L}(f) = \int f d\mu$  for all  $f \in \check{A}$ .  $\square$

**Remark 3.2.** In Corollary 3.1, one could easily show that for every compact subset  $K$  of  $X$ ,  $\chi_K$  is a uniform limit of a sequence in  $C_c(X)$  and hence  $C_c(X)$  is dense in  $C_c(X) + S_{\mathcal{B}(X)}$ . Therefore the measure obtained from lemma 2.3(4) has to be a Borel measure.

**Corollary 3.3** (Marshall [10]). *Suppose  $A$  is an  $\mathbb{R}$ -algebra,  $X$  is a Hausdorff space, and  $\hat{\cdot} : A \rightarrow C(X)$  is an  $\mathbb{R}$ -algebra homomorphism such that for some  $p \in A$ ,  $\hat{p} \geq 0$  on  $X$ , the set  $X_i = \hat{p}^{-1}([0, i])$  is compact for each  $i = 1, 2, \dots$ . Then for every linear functional  $L : A \rightarrow \mathbb{R}$  satisfying  $L(\text{Psd}_{\hat{A}}(X)) \subseteq [0, \infty)$ , there exists a Radon measure  $\mu$  on  $X$  such that*

$$\forall a \in A \quad L(a) = \int_X \hat{a} \, d\mu.$$

*Proof.* It suffices to show that  $\hat{A}$  is a  $C_c(X)$ -adapted space. Take  $g \in A$  with  $\hat{g} \geq 0$  on  $X$  and let  $f = g + p$ . The set  $X_n = \hat{f}^{-1}([0, n]) \subseteq \hat{p}^{-1}([0, n])$  is compact and  $X_n \subseteq X_{n+1}$ . Also for each  $n$ , the set  $Y_n = X_{n+1} \cap \hat{f}^{-1}([n + \frac{1}{2}, \infty))$  is closed and disjoint from  $X_n$ .

By Urysohn's lemma there exists a continuous function  $e_n : X_{n+1} \rightarrow [0, 1]$  such that  $e_n|_{Y_n} = 0$  and  $e_n|_{X_n} = 1$ . Extending  $e_n$  to  $X$  by defining  $e_n(x) = 0$  off  $X_{n+1}$ , we have  $g_n = \hat{g}e_n \in C_c(X)$ .

Fix  $\epsilon > 0$  and  $N > 0$  with  $\frac{1}{N} < \epsilon$ , we have

$$\hat{g} - g_N \leq \epsilon \hat{f}^2$$

on  $X$  or  $\hat{g} \leq \epsilon \hat{f}^2 + g_N$  and hence  $\hat{g} \preccurlyeq_{C_c(X)} \hat{f}^2$  as desired. Now, theorem 2.5 applies and the existence of such a measure follows.  $\square$

**3.1. Application to Classical Moment Problem.** One can apply Corollary 3.3 to give a short solution for the classical  $K$ -moment problem:

**Corollary 3.4** (Haviland [6, 7]). *Let  $K$  be a closed subset of  $\mathbb{R}^n$  and  $L : \mathbb{R}[X] \rightarrow \mathbb{R}$ . Then  $L$  admits an integral representation with respect to a positive Radon measure supported on  $K$  if and only if  $L(\text{Psd}(K)) \subseteq [0, \infty)$ .*

*Proof.* Take  $p = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X]$ , clearly  $p^{-1}([0, n])$  is compact for each  $n$ . Thus Corollary 3.3 applies.  $\square$

**3.2. Application to Truncated Moment Problem.** Let  $A$  be a subspace of  $C(X)$  and  $\mathfrak{B} \subset A$  a basis for  $A$ . Let  $V = A + \text{Span}\{1 + p^2 : p \in \mathfrak{B}\}$  and suppose that for every  $g \in A$  there exists  $f \in V$  such that for every  $\epsilon > 0$  there is  $h \in C_c(X)$  satisfying  $|g| \leq \epsilon|f| + h$ . Then according to Remark 2.6, any positive functional  $L : V \rightarrow \mathbb{R}$ , admits an integral representation with respect to a Borel measure on  $X$  over  $A$ . One can improve this situation in the case where  $A$  is a finite dimensional subspace of the polynomials  $\mathbb{R}[X]$ .

**Proposition 3.5** (Curto–Fialkow [3]). *Let  $L : \mathbb{R}[X]_{2d} \rightarrow \mathbb{R}$ ,  $d \geq 1$ , be a  $K$ -positive functional where  $K \subseteq \mathbb{R}^n$ . Then there exists a Radon measure  $\mu$  on  $K$  such that for all  $p \in \mathbb{R}[X]_{2d-1}$ ,  $L(p) = \int f \, d\mu$ .*

Here  $\mathbb{R}[X]_m$  denotes the set of all polynomials of degree at most  $m$ .

*Proof.* Every polynomial  $p \in \mathbb{R}[X]_{2d-1}$ , is bounded by a polynomial of degree  $2d$  outside of a compact set. In other words, there exists  $h \in \mathbb{R}[X]_{2d}$  and  $h \in C_c(K)$  such that  $|p| \leq |f| + h$ . Therefore Remark 2.6 applies and hence  $L|_{\mathbb{R}[X]_{2d-1}}$  is representable by a Radon measure on  $K$ .  $\square$

As it is proved in [3, Theorem 2.2], the combination of proposition 3.5 and Bayer–Teichmann [1, Theorem 2], implies the following:

**Corollary 3.6.** *A functional  $L : \mathbb{R}[X]_{2d} \rightarrow \mathbb{R}$  admits a  $K$ -representing measure if and only if  $L$  extends positively to  $\mathbb{R}[X]_{2d+2}$ .*

*Proof.* See [3, Theorem 2.2].  $\square$

**3.3. Representation of a Functional on  $C(X)$ .** Now we state a variation of Riesz–Markov–Kakutani representation theorem where  $C_c(X)$  is replaced by  $C(X)$  and  $X$  is assumed to be a  $\sigma$ -compact space. We show that a positive linear functional on  $C(X)$  is representable via a Radon measure on  $X$ , but it has to have a compact support.

**Definition 3.7.** A subalgebra  $B \subseteq A$  is called  $\sigma$ -complete when for any sequence  $(b_n)_{n \in \mathbb{N}} \subset B$ , if  $b = \sum_n b_n$  exists in  $A$  then  $b \in B$ .

**Theorem 3.8.** *Let  $A$  be a  $\sigma$ -complete subalgebra of  $C(X)$  which separates points of  $X$ . Suppose that  $X$  is a  $\sigma$ -compact, locally compact and Hausdorff space and  $L : A \rightarrow \mathbb{R}$  is a positive linear functional. Then there exists a Radon measure  $\mu$  on  $X$  with compact support, such that  $L(f) = \int_X f \, d\mu$  for all  $f \in A$ .*

*Proof.* We break the proof into three steps.

**Step 1.** There is a sequence  $(X_n)_{n \in \mathbb{N}}$  of compact subsets of  $X$  such that  $X = \bigcup_n X_n$  and  $X_n \subseteq X_{n+1}^\circ$ .

To see this, choose a countable compact cover  $(C_n)_{n \in \mathbb{N}}$  for  $X$  and set  $D_n = \bigcup_{i=1}^n C_i$ . For every  $n$ , the set  $D_n$  is compact and since  $X$  is locally compact, every  $x \in D_n$  has a neighbourhood  $V_x$  where  $\overline{V_x}$  is compact. The collection  $\{V_x : x \in D_n\}$  is an open cover for  $D_n$  and hence there are  $x_1, \dots, x_t \in D_n$  such that  $D_n \subseteq \bigcup_{i=1}^t V_{x_i}$ . Now, let  $X_n = \bigcup_{i=1}^t \overline{V_{x_i}}$ , the collection  $(X_n)_{n \in \mathbb{N}}$  is the desired sequence.

**Step 2.** There exists a compact  $K \subseteq X$  such that  $f|_K = 0$  implies  $L(f) = 0$ .

Assume that such  $K$  does not exist. So for every compact  $C \subseteq X$ , there exists  $f_C \in A$  such that  $f_C|_C = 0$  but  $L(f_C) \neq 0$ . Substitute  $f_C$  with  $f_C^2$  if necessary, we can assume that  $f_C \geq 0$  and hence  $L(f_C) > 0$ . Thus for sufficiently large  $N_C$ ,  $L(N_C f_C) > 1$ . Take a compact cover  $(X_n)_{n \in \mathbb{N}}$  of  $X$  with  $X_n \subseteq X_{n+1}^\circ$  and denote  $N_{X_n} f_{X_n}$  by  $g_n$ . For every  $x \in X$ ,  $\sum_n g_n(x)$  is finite and is continuous according to the choice of  $(X_n)_{n \in \mathbb{N}}$ . Therefore  $g = \sum_n g_n \in C(X)$  and since  $A$  is  $\sigma$ -complete,  $g \in A$ . Computing  $L(g)$ , we get

$$\sum_n 1 < \sum_n L(g_n) \leq L\left(\sum_n g_n\right) = L(g) < \infty,$$

a contradiction. This proves that such a compact  $K \subseteq X$  must exist.

**Step 3.** There exists a linear functional  $\tilde{L}$  on  $C(K)$  such that

$$\forall f \in A \quad L(f) = \tilde{L}(f|_K).$$

Let  $I(K) = \{f \in C(X) : f|_K = 0\}$  and  $I_A(K) = I(K) \cap A$ . Applying Titzte extension theorem, we can identify  $\frac{C(X)}{I(K)}$  with  $C(K)$ . By Step 2,  $I_A(K) \subseteq \ker L$ , therefore  $L$  factors through  $\pi : A \rightarrow \frac{A}{I_A(K)}$ . Moreover the map  $\vartheta : \frac{A}{I_A(K)} \rightarrow \frac{C(X)}{I(K)}$  defined by  $f + I_A(K) \mapsto f + I(K)$  is well-defined and injective. Since  $A$  separates points of  $X$ ,  $\frac{A}{I_A(K)}$  also separates points of  $K$  and by Stone-Weierstrass theorem, image of  $\vartheta$  is dense in  $C(K)$ . Hence  $\bar{L}$  admits an extension  $\tilde{L} : C(K) \rightarrow \mathbb{R}$  which respects positivity and  $\tilde{L}(f|_K) = L(f)$  for all  $f \in A$  (see Figure 1). Now Riesz–Markov–Kakutani representation theorem guarantees the existence of a



$$\begin{array}{ccccc}
 A & \xrightarrow{\pi} & \frac{A}{I_A(K)} & \xrightarrow{\vartheta} & \frac{C(X)}{I(K)} \\
 & \searrow L & \downarrow \tilde{L} & \swarrow \tilde{L} & \\
 & & \mathbb{R} & & 
 \end{array}$$

FIGURE 1.

Radon measure  $\nu$  on  $K$  such that

$$\forall g \in C(K) \quad \tilde{L}(g) = \int g \, d\nu.$$

Extend  $\nu$  to a measure  $\mu$  on  $X$  by defining  $\mu(E) = \nu(E \cap K)$ , we have

$$\forall f \in A \quad L(f) = \int f \, d\mu,$$

as desired.  $\square$

#### 4. REPRESENTATION OF A FUNCTIONAL OVER A SUBALGEBRA OF $\ell^\infty(X)$

In this section we consider a functional  $L : A \rightarrow \mathbb{R}$  where  $A$  is a unital commutative  $\mathbb{R}$ -algebra for which there exists an  $\mathbb{R}$ -algebras homomorphism  $\iota : A \rightarrow \ell^\infty(X)$ , and investigate the possibility of representing  $L$  as an integral with respect to a measure  $\mu$  on  $X$  such that

$$L(a) = \int_X \iota a \, d\mu \quad \forall a \in A.$$

The algebra  $\ell^\infty(X)$  is naturally equipped with a norm defined as

$$\|f\|_X = \sup_{x \in X} |f(x)|$$

for every  $f \in \ell^\infty(X)$ . This induces a seminorm  $\|\cdot\|$  on  $A$  defined by  $\|a\| = \|\iota a\|_X$ . The relation between locally multiplicatively convex topologies (such as  $\|\cdot\|_X$ ), positive cones and integral representation of continuous functionals has been studied in [5]. What follows, is a consequence of [5, Theorem 3.7].

Let  $d \geq 1$  an integer number. We denote by  $\sum A^{2d}$ , the cone of all finite sums of  $2d^{th}$  powers of elements of  $A$ ; i.e.,

$$\sum A^{2d} = \{a_1^{2d} + \cdots + a_m^{2d} : a_1, \dots, a_m \in A, m \geq 1\}.$$

We denote the set of all  $\|\cdot\|$ -continuous real valued algebra homomorphisms on  $A$  by  $\mathbf{sp}_{\|\cdot\|}(A)$  which is known as the Gelfand spectrum of the seminormed algebra  $(A, \|\cdot\|)$ .

**Proposition 4.1.** *Let  $\iota : A \rightarrow \ell^\infty(X)$  be an  $\mathbb{R}$ -algebras homomorphism and  $d \geq 1$  an integer. Then  $\overline{\sum A^{2d}}^{\|\cdot\|_X} = \text{Psd}_A(X)$  where  $\|\cdot\|$  is the pull-back seminorm induced by  $\|\cdot\|_X$  on  $A$  through  $\iota$ .*

*Proof.* The map  $\iota$  induces a function  $\iota_* : X \rightarrow \mathbf{sp}_{\|\cdot\|}(A)$  by  $\iota_*(x)(a) = \iota a(x)$ .

*Claim.*  $\iota_* X$  is dense in  $\mathbf{sp}_{\|\cdot\|}(A)$ .

Every element  $a \in A$  corresponds to a continuous map  $\hat{a} : \mathbf{sp}_{\|\cdot\|}(A) \rightarrow \mathbb{R}$  defined as  $\hat{a}(\alpha) = \alpha(a)$ . If  $\iota_* X$  is not dense in  $\mathbf{sp}_{\|\cdot\|}(A)$ , then we can take  $\alpha \in \mathbf{sp}_{\|\cdot\|}(A) \setminus \overline{\iota_* X}$ .

Since  $\mathfrak{sp}_{\|\cdot\|}(A)$  is compact and Hausdorff, there exists  $f \in C(\mathfrak{sp}_{\|\cdot\|}(A))$  such that  $f(\alpha) = 1$  and  $f|_{\iota_* X} = 0$ . Clearly,  $A$  separates points of  $\mathfrak{sp}_{\|\cdot\|}(A)$ , by Stone–Weierstrass theorem it is dense in  $C(\mathfrak{sp}_{\|\cdot\|}(A))$ . Therefore, for  $\epsilon > 0$ , there exists  $a_\epsilon \in A$  with  $\|f - a_\epsilon\| < \epsilon$ . Take  $\epsilon > 0$  such that  $\frac{1-\epsilon}{\epsilon} > 1$ . Then  $|f(\alpha - \alpha(a_\epsilon))| = |1 - \alpha(a_\epsilon)| < \epsilon$  or  $1 - \epsilon < |\alpha(a_\epsilon)| < 1 + \epsilon$ . Also  $|f(\iota_* x) - \iota a_\epsilon(x)| = |\iota a_\epsilon(x)| < \epsilon$  for all  $x \in X$ . But

$$\begin{aligned} \sup_{\beta \in \mathfrak{sp}_{\|\cdot\|}(A)} |\beta(a_\epsilon)| &\leq \|a_\epsilon\| \\ &\leq \sup_{x \in X} |\iota a_\epsilon(x)| \\ &\leq \epsilon \\ &< 1 - \epsilon, \end{aligned}$$

and hence  $|\alpha(a_\epsilon)| < 1 - \epsilon$ , a contradiction. This proves the claim.

By [5, Theorem 3.7],  $\overline{\sum A^{2d}}^{\|\cdot\|} = \text{Psd}_A(\mathfrak{sp}_{\|\cdot\|}(A))$ . By the above claim  $\iota_* X$ , is dense in  $\mathfrak{sp}_{\|\cdot\|}(A)$  and therefore  $\hat{a} \geq 0$  on  $\mathfrak{sp}_{\|\cdot\|}(A)$  if and only if  $\hat{a} \geq 0$  on  $\iota_* X$  or equivalently  $\iota a \geq 0$  on  $X$ .  $\square$

An application of Banach separation theorem shows that  $\overline{\sum A^{2d}}^{\|\cdot\|^x} = \text{Psd}_A(X)$  holds, if and only if every  $\|\cdot\|$ -continuous functional  $L : A \rightarrow \mathbb{R}$  that is non-negative on  $\sum A^{2d}$ , admits an integral representation with respect to a Radon measure  $\mu$  supported on  $\mathfrak{sp}_{\|\cdot\|}(A)$  ([5, Corollary 3.8]), such that:

$$L(a) = \int_{\mathfrak{sp}_{\|\cdot\|}(A)} \hat{a} \, d\mu.$$

Since we are representing  $A$  as a subalgebra of  $\ell^\infty(X)$ , it is natural to ask if  $L$  comes from a positive measure  $\nu$  on  $X$  such that  $L(a) = \int_X \iota a \, d\nu$ .

The answer to the above question is negative in general. We consider the case where  $A$  separates points of  $X$  and hence  $\mathfrak{sp}_{\|\cdot\|}(A)$  contains  $X$  as a dense subspace, according to the claim proved in proposition 4.1.

**Example 4.2.** Suppose that  $X$  is equipped with a non-compact completely regular topology and  $A = C_b(X)$ . Then  $\beta X \setminus X \neq \emptyset$ . Every homomorphism  $\alpha \in \beta X \setminus X$  is a continuous functional which corresponds to the point-measure  $\delta_\alpha$  supported at  $\alpha$  itself. Clearly  $\alpha(f) = \int f \, d\delta_\alpha$ , and it is not representable with respect to any measure supported on  $X$ .

The next example shows that there might exist a non-extremal functional which is representable by a measure on  $\mathfrak{sp}_{\|\cdot\|}(A)$  but not on  $X$ .

**Example 4.3.** Suppose that  $Y$  is a separable compact Hausdorff space and  $\mu$  be a measure on  $Y$  such that each point is of zero measure. Let  $X$  be a countable dense subset of  $Y$  and  $A = C_b(X)$ . Then  $\mathfrak{sp}(A) = Y$  and the functional  $L(f) = \int_Y f \, d\mu$  is positive and continuous. But the restriction of  $\mu$  on  $X$  is the 0 measure. Therefore  $L$  does not have an integral representation with respect to a measure on  $X$ .

**Remark 4.4.**

- (1) For any positive Radon measure  $\mu$  on  $\mathfrak{sp}_{\|\cdot\|}(A)$ , the corresponding functional  $L_\mu$  defined as  $L_\mu(a) = \int \hat{a} \, d\mu$  is  $\|\cdot\|$ -continuous. Since  $\mathfrak{sp}_{\|\cdot\|}(A)$  is compact,  $L_\mu(1) = \mu(\mathfrak{sp}_{\|\cdot\|}(A)) \leq \infty$ , and  $L_\mu(a) \leq \|a\| L_\mu(1)$  which is equivalent to  $\|\cdot\|$ -continuity of  $L_\mu$ .

- (2) Let  $X$  be a compact Hausdorff space,  $A \subseteq C(X)$  a unital separating subalgebra and  $\mu$  and  $\nu$  radon measures on  $X$  such that  $\int a \, d\mu = \int a \, d\nu$  for all  $a \in A$ . Then one can prove that  $\mu = \nu$ , almost everywhere.

To see this, we first prove that  $\text{supp}(\mu) = \text{supp}(\nu)$ . Otherwise, there exists a compact set  $Z \subseteq X \setminus \text{supp}(\mu)$  with  $\nu(Z) > 0$ . We can choose  $\epsilon$  such that  $0 < \epsilon < \frac{\nu(Z)}{\mu(X) + \nu(Z)}$ . Since  $\text{supp}(\mu)$  and  $Z$  are compact, there exists a continuous function on  $X$  such that  $f|_Z = 1$  and  $f|_{\text{supp}(\mu)} = 0$ . By Stone-Weierstrass theorem,  $\exists a \in A$  such that  $|f - a| \leq \epsilon$  on  $Z$  and  $|a| \leq \epsilon$  on  $\text{supp}(\mu)$ . Replacing  $a$  by  $a^2$  if necessary, we can assume that  $a \geq 0$ . Therefore  $\int a \, d\mu \leq \epsilon\mu(X)$  and

$$(1 - \epsilon)\nu(Z) \leq \int_Z a \, d\nu \leq \int a \, d\nu.$$

Thus  $\nu(Z) \leq (\mu(\text{supp}(\mu)) + \nu(Z))\epsilon$ , a contradiction. Similarly, we can prove that  $\int g \, d\mu = \int g \, d\nu$  for all  $g \in C(\text{supp}(\mu))$  and hence  $\mu = \nu$ , almost everywhere.

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